Smoothing and Approximation

Convolution is particularly useful for smoothing rough functions.

Prop 1. Let $f \in L^1$, $g \in C^k$, and assume $\partial^\alpha g$ bdd, $\forall |\alpha| \le k$. Then, $f * g \in C^k$ and $\partial^\alpha (f*g) = f * \partial^\alpha g$.

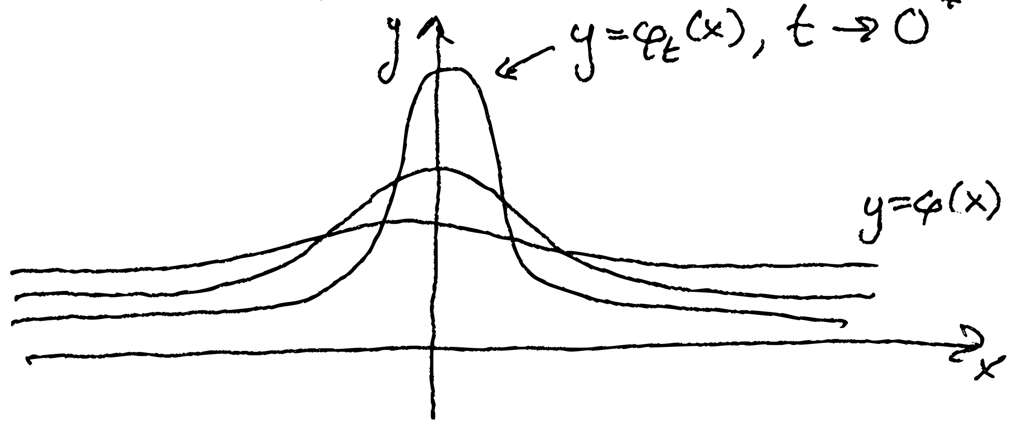
Pf. Since each $|\partial^\alpha g| \le C$, we have
$$|f(y)|\,|\partial_x^\alpha g(x-y)| \le C|f| \in L^1.$$
Thus, we may differentiate $f*g$ under integral sign $\partial^\alpha \int = \int \partial^\alpha$. We get
$$\partial^\alpha (f*g) = \partial_x^\alpha \int f(y)\, g(x-y)\, dy =$$
$$\int f(y)\, \partial_x^\alpha g(x-y)\, dy = \int f(y)\, (\partial^\alpha g)(x-y)\, dy$$
$= f * \partial^\alpha g$. Continuity follows similarly.

$\boxtimes$

Rem. The typical application of Prop 1 is to use $g \in C_c^\infty$ or $g \in \mathcal{S}$.

Once you have smoothed a function f, you would also like to have a way of recovering f (or info about f).

For this, let $\varphi : \mathbb{R}^n \to \mathbb{R}$, $t > 0$ and let $\varphi_t(x) = \frac{1}{t^n}\varphi(x/t)$.



Note:

$$\int \varphi_t(x)\,dx = \int \frac{1}{t^n}\varphi(x/t)\,dx = \left\{\begin{matrix} y = x/t \\ dx = t^n dy \end{matrix}\right\} = \int \varphi(y)\,dy$$

Thm 1. Let $\varphi \in L^1$, $\int \varphi = 1$.

(i) If $f \in L^p$, then $f * \varphi_t \to f$ in L^p $(1 \le p < \infty)$

(ii) If $f \in L^\infty$ + unif. cont., then $f * \varphi_t \to f$ uniformly.

(iii) If $f \in L^\infty$ + cont. on $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, then $f * \varphi_t \to f$ uniformly on compacts $K \subset\subset U$.

Pf: (i) & (ii) follows from Minkowski's Ineq. + Prop 8.5. For $1 = \int \varphi_t(y)\,dy \Rightarrow$

$$|(f*\varphi_t)(x) - f(x)| = \left|\int [f(x-y)\varphi_t(y) - f(x)\varphi_t(y)]\,dy\right|$$

$$\le \int |f(x-y) - f(x)| \cdot |\varphi_t(y)|\,dy = \left\{ \begin{matrix} z = y/t \\ dy = t^n dz \end{matrix} \right\}$$

$$= \int |f(x-tz) - f(x)|\, |\varphi(z)|\,dz =$$

$= \int |(\tau_{tz} f)(x) - f(x)|\, |\varphi(z)|\, dz$ $\Big\}$ $1 \le p \le \infty$ in (i) or (ii).

Now, for each z, $|(\tau_{tz} f)(x) - f(x)| \le |(\tau_{tz} f)(x)| + |f(x)|$ and $\tau_{tz} f, f \in L^p$.

Moreover, $z \to \|(\tau_{tz} f) - f\|_{L^p} \le 2\|f\|_{L^p}$ and constants are in $L^1(\mathbb{R}^n, |\varphi| dm)$ since $\varphi \in L^1$. Thus, we may apply Minkowski's Ineq.: $(1 \le p \le \infty)$

(*) $\|f * \varphi_t - f\|_{L^p} \le \int \|(\tau_{tz} f) - f\|_{L^p} |\varphi(z)|\, dz$

For (i), $\|(\tau_{tz} f) - f\|_{L^p} \to 0$ as $t \to 0$, so, by dominated convergence, (*) $\Rightarrow$ $\|f * \varphi_t - f\|_{L^p} \to 0$.

For (ii), with $p = \infty$ above + $\|\cdot\|_{L^\infty} = \|\cdot\|_u$, unif. cont. $\Rightarrow \|(\tau_{tz} f) - f\|_u \to 0$ as $t \to 0$ $\underset{(*) + \text{dom. conv.}}{\Rightarrow}$ $\|f * \varphi_t - f\|_u \to 0$, $t \to 0$.

For (iii), we note that if $K \subset\subset \bar{U}$, and $K' \subset\subset \mathbb{R}^n$, then $\exists\, \delta > 0$, $K'' \subset\subset \bar{U}$ s.t. $x - tz \in K''$, $\forall x \in K, z \in K', |t| < \delta$.

Pick $\varepsilon > 0$. Since $f \in C(K'')$ f is unif. cont. on K'' $\Rightarrow \exists\, 0 < \delta' < \delta$ s.t.

$$\sup_{\substack{x \in K, z \in K' \\ |t| < \delta'}} |(\tau_{tz} f)(x) - f(x)| < \varepsilon .$$

Moreover, since $\varphi \in L^1$, we can apriori choose $K' \subset\subset \mathbb{R}^n$ s.t. $\int_{\mathbb{R}^n \setminus K'} |\varphi| < \varepsilon$. $\Rightarrow$ For $|t| < \delta'$,

$$\sup_{x \in K} |(f * \varphi_t)(x) - f(x)| \le \sup_{x \in K} \int |(\tau_{tz} f)(x) - f(x)| \cdot |\varphi(z)|\, dz$$

$$\le \sup_{x \in K} \int_{K'} |\tau_{tz} f(x) - f(x)|\, |\varphi(z)|\, dz \; +$$

$$\sup_{x \in K} \int_{\mathbb{R}^n \setminus K'} |\tau_{tz} f(x) - f(x)|\, |\varphi(z)|\, dz$$

$$< \varepsilon \cdot \|\varphi\|_{L^1} + 2\|f\|_{L^\infty} \cdot \varepsilon \Rightarrow \text{(iii)}$$

$\blacksquare$

Thm 2. Let $\varphi \in L^1$ s.t. for some $\alpha > 0$

$$|\varphi(x)| \le C(1+|x|)^{-(n+\alpha)}.$$

and $\int \varphi(x)\,dx = 1$. If $f \in L^p$, $1 \le p \le \infty$

then $f * \varphi_t \to f$, $\forall x \in L_f$

Lebesgue set of f

Pf uses "dyadic decomposition".

Pf. Let x be a Lebesgue pt, i.e.

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\,dy = 0$$

$$\Rightarrow \lim_{r\to 0} \frac{1}{m(B(0,r))} \int_{B(0,r)} |f(x-y) - f(x)|\,dy = 0$$

Since $m(B(0,r)) = C_n r^n \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ s.t

$$\int_{|y|<r} |f(x-y) - f(x)|\,dy \lesssim \varepsilon r^n, \quad \forall r \le \delta.$$

Consider, for given $\varepsilon > 0$ ($\longrightarrow \delta > 0$),

$$|(f * \varphi_t)(x) - f(x)| \le \int_{\mathbb{R}^n} |f(x-y) - f(x)|\, |\varphi_t(y)|\, dy$$

$$= \underbrace{\int_{|y|<\delta} |f(x-y) - f(x)| \cdot |\varphi_t(y)|\, dy}_{I_1 = I_1^{\delta}(t)} + \underbrace{\int_{|y|\ge\delta} \text{same}}_{I_2 = I_2^{\delta}(t)}$$

① Estimate I_2 first (easier):

$$I_2 = \int_{|y|\ge\delta} |f(x-y) - f(x)|\, |\varphi_t(y)|\, dy \le \left\{ \begin{matrix} \text{Hölder} \\ + \\ \Delta\text{-ineq.} \end{matrix} \right\}$$

$$\le \|f\|_{L^p} \|\chi \varphi_t\|_{L^{p^*}} + |f(x)|\, \|\chi \varphi_t\|_{L^1} \qquad \tfrac{1}{p} + \tfrac{1}{p^*} = 1$$

So, suffices to show $\|\chi \varphi_t\|_{L^q} \to 0$ for $1 \le q \le \infty$.

Recall: $|\varphi(x)| \le C(1+|x|)^{-(n+\alpha)}$, $\alpha > 0$.

- If $q=\infty$, then for $|y| \ge \delta$

$$|\varphi_t(y)| = \frac{1}{t^n}|\varphi(y/t)| \le \frac{C}{t^n}\frac{1}{(1+|y/t|)^{n+\alpha}} \le \frac{C}{(t+\delta)^n}\frac{1}{(1+\delta/t)^\alpha}$$

$\Rightarrow \|\chi\varphi_t\|_{L^\infty} \to 0$ as $t \to 0$.

- If $1 \le q < \infty$, then

$$\|\chi\varphi_t\|_{L^q}^q = \int_{|y|\ge\delta} \frac{1}{t^{nq}}|\varphi(y/t)|^q dy =$$

$$= \int_{|z|\ge\delta/t} \frac{1}{t^{n(q-1)}}|\varphi(z)|^q dz \le \frac{C^q}{t^{n(q-1)}}\int_{|z|\ge\delta/t}\frac{dz}{(1+|z|)^{(n+\alpha)q}}$$

$$\le \frac{C'}{t^{n(q-1)}}\int_{\delta/t}^{\infty}\frac{r^{n-1}dr}{r^{(n+\alpha)q}} =$$

$$\frac{1}{r^{(q-1)n+q\alpha+1}}$$

$$= \frac{C''}{t^{n(q-1)}} \cdot \frac{1}{(\delta/t)^{(q-1)n+n\alpha}} = C_\delta\, t^{n\alpha} \to 0$$

δ fixed! as $t \to 0$.

(2) Estimate $I_1 - I_1^\delta(t)$ by dyadic decomp.

<u>Recall</u>: $I_1 = \int_{|y|<\delta} |f(x-y) - f(x)|\,|\varphi_t(y)|\,dy$

and δ is chosen s.t. $\forall\, 0<r<\delta$,

$$\int_{|y|<r} |f(x-y) - f(x)|\,dy \le \varepsilon r^n.$$

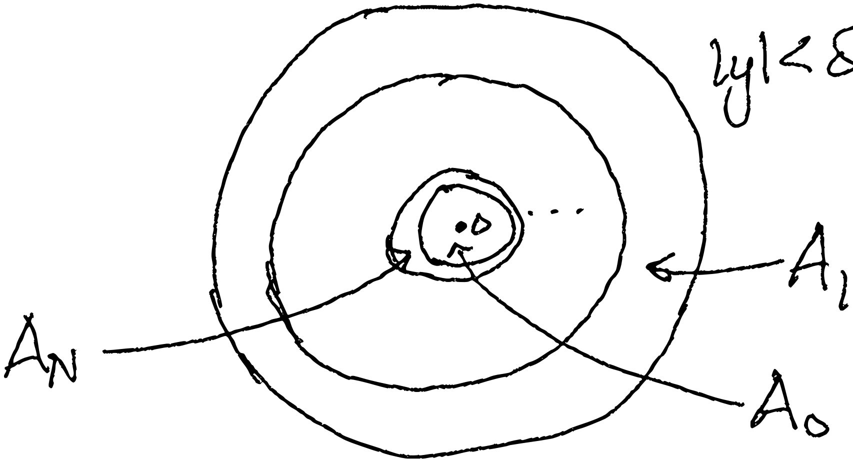
First, fix $t>0$ so small that $\delta/t \ge 1$ and decompose the ball $B(0,\delta) = \{|y|<\delta\}$

$$B(0,\delta) = \bigcup_{k=0}^{N} A_k,$$

where $2^N \le \delta/t \le 2^{N+1}$ and

$A_0 = \{ |y| < \delta 2^{-N} \}$ and for $k \geq 1$

$A_k = \{ \delta 2^{-k} \leq |y| < \delta 2^{-k+1} \}$.



On A_k, $1 \leq k \leq N$, we have

$$|\varphi_t(y)| \leq \frac{C}{t^n} \frac{1}{(1+|y/t|)^{n+\alpha}} \leq C \frac{1}{t^n} \left(\frac{\delta}{t} 2^{-k} \right)^{-(n+\alpha)}$$

$\left|\frac{y}{t}\right| \geq \frac{\delta}{t} 2^{-k} \geq 2^{N-k} \geq 1$

and on A_0

$$|\varphi_t(y)| \leq \frac{C}{t^n}$$

Now, we have

$$\int_{|y|<\delta} |f(x-y)-f(x)|\,|\varphi_t(y)|\,dy = \sum_{k=0}^{N} \int_{A_k} \text{"same"}\, dy$$

$$\leq \frac{C}{t^n}\Bigg(\int_{|y|<\delta 2^{-N}} |f(x-y)-f(x)|\,dy +$$

$$\sum_{k=1}^{N} \left(\frac{\delta}{t} 2^{-k}\right)^{-(n+\alpha)} \int_{\delta 2^{-k} \leq |y| \leq \delta 2^{-k+1}} |f(x-y)-f(x)|\,dy \Bigg)$$

$$\leq \frac{C}{t^n}\left(\varepsilon(\delta 2^{-N})^n + \sum_{k=1}^{N} \left(\frac{\delta}{t} 2^{-k}\right)^{-(n+\alpha)} \varepsilon(\delta 2^{-k+1})^n \right)$$

$$= \varepsilon C \left(\left(\frac{\delta}{t} 2^{-N}\right)^n + \sum_{k=1}^{N} 2^n \left(\frac{\delta}{t} 2^{-k}\right)^{-\alpha} \right)$$

$$\leq \left\{ 2^N \leq \frac{\delta}{t} \leq 2^{N+1} \right\} \leq$$

$$\varepsilon C \left(2^n + 2^n \sum_{k=1}^{N} \left(2^{N-k}\right)^{-\alpha} \right) \leq$$

$$\varepsilon 2^n C \left(1 + \sum_{j=0}^{\infty} (2^{-\alpha})^j \right) \leq C_{n,\alpha}$$

depends only on n,α.

fixed, $0<t<\delta$

i.e. $I_1^\delta(t) \leq C_{n,\alpha} \cdot \varepsilon$, where $\varepsilon>0$ arbitrary.

In other words, if $x \in L_f$ $\forall \varepsilon>0$ $\exists \delta>0$ s.t. $\forall\, 0<t<\delta$

$$|(f*\varphi_t)(x) - f(x)| \leq I_1^\delta(t) + I_2^\delta(t) \leq$$

$$\leq C_n \varepsilon + I_2^\delta(t) \longrightarrow C_n\varepsilon \text{ as } t \to 0.$$

Thus, we have

$$\lim_{t\to 0} |(f*\varphi_t)(x) - f(x)| = 0.$$

$\blacksquare$

Easy consequences of the smoothing and approximation results above (Pfs are DIY):

<u>$\mathcal{C}_c^\infty$ approx. in $L^p / \mathcal{C}_0$.</u> $\mathcal{C}_c^\infty$ is dense in L^p and $\mathcal{C}_0$.

<u>$\mathcal{C}_c^\infty$-Urysohn.</u> Let $K \subset\subset U \subset \mathbb{R}^n$ w/ K cpct, U open. Then $\exists$ $\varphi \in \mathcal{C}_c^\infty$ s.t. $0 \le \varphi \le 1$, $\varphi = 1$ on K, and $\operatorname{supp} \varphi \subset\subset U$.